Exercise 15D: Solutions

1 a $\det(\mathbf{A}) = 2 \times 2 - 1 \times 3$
$= 1$

b $\mathbf{A}^{-1} = \frac{1}{1}\begin{bmatrix} 2 & -1 \\ -3 & 2 \end{bmatrix} = \begin{bmatrix} 2 & -1 \\ -3 & 2 \end{bmatrix}$

c $\det(\mathbf{B}) = -2 \times 2 - -2 \times 3$
$= 2$

d $\mathbf{B}^{-1} = \frac{1}{2}\begin{bmatrix} 2 & 2 \\ -3 & -2 \end{bmatrix}$

2 a $\text{Determinant} = 3 \times -1 - -1 \times 4 = 1$

$\mathbf{A}^{-1} = \frac{1}{1}\begin{bmatrix} -1 & 1 \\ -4 & 3 \end{bmatrix} = \begin{bmatrix} -1 & 1 \\ -4 & 3 \end{bmatrix}$

b $\text{Determinant} = 3 \times 4 - 1 \times -2 = 14$

$$\mathbf{A}^{-1} = \frac{1}{14}\begin{bmatrix} 4 & -1 \\ 2 & 3 \end{bmatrix} = \begin{bmatrix} \frac{2}{7} & -\frac{1}{14} \\ \frac{1}{7} & \frac{3}{14} \end{bmatrix}$$

c $\text{Determinant} = 1 \times k - 0 \times 0 = k$

$\mathbf{A}^{-1} = \frac{1}{k}\begin{bmatrix} k & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & \frac{1}{k} \end{bmatrix}$

d $\text{Determinant} = \cos\theta \times \cos\theta - -\sin\theta \times \sin\theta$
$= 1$

since $\cos^2\theta + \sin^2\theta = 1$

$$\mathbf{A}^{-1} = \frac{1}{1}\begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$$

3 Suppose $\mathbf{AB} = \mathbf{BA} = \mathbf{I}$
and $\mathbf{AC} = \mathbf{CA} = \mathbf{I}$
Then
$\mathbf{C} = \mathbf{CI} = \mathbf{C}(\mathbf{AB}) = (\mathbf{CA})\mathbf{B} = \mathbf{IB} = \mathbf{B}$

4 a $\det(\mathbf{A}) = 2 \times -1 - 1 \times 0 = -2$

$$\mathbf{A}^{-1} = \frac{1}{-2}\begin{bmatrix} -1 & -1 \\ 0 & 2 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ 0 & -1 \end{bmatrix}$$

$\det(\mathbf{B}) = 1 \times 1 - 0 \times 3 = 1$

$$\mathbf{B}^{-1} = \frac{1}{1}\begin{bmatrix} 1 & 0 \\ -3 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ -3 & 1 \end{bmatrix}$$

b $\mathbf{AB} = \begin{bmatrix} 2 & 1 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 3 & 1 \end{bmatrix}$

$$= \begin{bmatrix} 2\times 1 + 1\times 3 & 2\times 0 + 1\times 1 \\ 0\times 1 + -1\times 3 & 0\times 0 + -1\times 1 \end{bmatrix}$$

$$= \begin{bmatrix} 5 & 1 \\ -3 & -1 \end{bmatrix}$$

$\det(\mathbf{AB}) = 5\times -1 - 1\times -3 = -2$

$$(\mathbf{AB})^{-1} = \frac{1}{-2}\begin{bmatrix} -1 & -1 \\ 3 & 5 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ -\frac{3}{2} & -\frac{5}{2} \end{bmatrix}$$

c $\mathbf{A}^{-1}\mathbf{B}^{-1} = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -3 & 1 \end{bmatrix}$

$$= \begin{bmatrix} \frac{1}{2}\times 1 + \frac{1}{2}\times -3 & \frac{1}{2}\times 0 + \frac{1}{2}\times 1 \\ 0\times 1 + -1\times -3 & 0\times 0 + -1\times 1 \end{bmatrix}$$

$$= \begin{bmatrix} -1 & \frac{1}{2} \\ 3 & -1 \end{bmatrix}$$

$$\mathbf{B}^{-1}\mathbf{A}^{-1} = \begin{bmatrix} 1 & 0 \\ -3 & 1 \end{bmatrix} \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ 0 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} 1\times\frac{1}{2} + 0\times 0 & 1\times\frac{1}{2} + 0\times -1 \\ -3\times\frac{1}{2} + 1\times 0 & -3\times\frac{1}{2} + 1\times -1 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ -\frac{3}{2} & -\frac{5}{2} \end{bmatrix}$$

$$(\mathbf{AB})^{-1} = \mathbf{B}^{-1}\mathbf{A}^{-1}$$

5 a $\det(\mathbf{A}) = 4\times 1 - 3\times 2 = -2$

$$\mathbf{A}^{-1} = \frac{1}{-2}\begin{bmatrix} 1 & -3 \\ -2 & 4 \end{bmatrix}$$

$$= \begin{bmatrix} -\frac{1}{2} & \frac{3}{2} \\ 1 & -2 \end{bmatrix}$$

b If $\mathbf{AX} = \begin{bmatrix} 3 & 4 \\ 1 & 6 \end{bmatrix}$, multiply both sides from the left by $\mathbf{A}^{-1}$.

$$\mathbf{A}^{-1}\mathbf{AX} = \mathbf{A}^{-1}\begin{bmatrix} 3 & 4 \\ 1 & 6 \end{bmatrix}$$

$$\therefore \mathbf{IX} = \mathbf{X}$$

$$= \begin{bmatrix} -\frac{1}{2} & \frac{3}{2} \\ 1 & -2 \end{bmatrix} \begin{bmatrix} 3 & 4 \\ 1 & 6 \end{bmatrix}$$

$$= \begin{bmatrix} -\frac{1}{2}\times 3 + \frac{3}{2}\times 1 & -\frac{1}{2}\times 4 + \frac{3}{2}\times 6 \\ 1\times 3 + -2\times 1 & 1\times 4 + -2\times 6 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 7 \\ 1 & -8 \end{bmatrix}$$

c If $\mathbf{YA} = \begin{bmatrix} 3 & 4 \\ 1 & 6 \end{bmatrix}$, multiply both sides from the right by $\mathbf{A}^{-1}$.

$$\mathbf{YAA}^{-1} = \begin{bmatrix} 3 & 4 \\ 1 & 6 \end{bmatrix} \mathbf{A}^{-1}$$

$$\therefore \mathbf{YI} = \mathbf{Y}$$

$$= \begin{bmatrix} 3 & 4 \\ 1 & 6 \end{bmatrix} \begin{bmatrix} -\frac{1}{2} & \frac{3}{2} \\ 1 & -2 \end{bmatrix}$$

$$= \begin{bmatrix} 3 \times -\frac{1}{2} + 4 \times 1 & 3 \times \frac{3}{2} + 4 \times -2 \\ 1 \times -\frac{1}{2} + 6 \times 1 & 1 \times \frac{3}{2} + 6 \times -2 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{5}{2} & -\frac{7}{2} \\ \frac{11}{2} & -\frac{21}{2} \end{bmatrix}$$

6 a If $\mathbf{AX} + \mathbf{B} = \mathbf{C}$ then $\mathbf{AX} = \mathbf{C} - \mathbf{B}$

$$\therefore \mathbf{AX} = \begin{bmatrix} 3 & 4 \\ 2 & 6 \end{bmatrix} - \begin{bmatrix} 4 & -1 \\ 2 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} -1 & 5 \\ 0 & 4 \end{bmatrix}$$

$$\det(\mathbf{A}) = 3 \times 6 - 2 \times 1 = 16$$

$$\mathbf{A}^{-1} = \frac{1}{16} \begin{bmatrix} 6 & -2 \\ -1 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{3}{8} & -\frac{1}{8} \\ -\frac{1}{16} & \frac{3}{16} \end{bmatrix}$$

If $\mathbf{AX} = \begin{bmatrix} -1 & 5 \\ 0 & 4 \end{bmatrix}$, multiply both sides from the left by $\mathbf{A}^{-1}$.

$$\mathbf{A}^{-1}\mathbf{AX} = \mathbf{A}^{-1} \begin{bmatrix} -1 & 5 \\ 0 & 4 \end{bmatrix}$$

$$\therefore \mathbf{IX} = \mathbf{X}$$

$$= \begin{bmatrix} \frac{3}{8} & -\frac{1}{8} \\ -\frac{1}{16} & \frac{3}{16} \end{bmatrix} \begin{bmatrix} -1 & 5 \\ 0 & 4 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{3}{8} \times -1 + -\frac{1}{8} \times 0 & \frac{3}{8} \times 5 + -\frac{1}{8} \times 4 \\ -\frac{1}{16} \times -1 + \frac{3}{16} \times 0 & -\frac{1}{16} \times 5 + \frac{3}{16} \times 4 \end{bmatrix}$$

$$= \begin{bmatrix} -\frac{3}{8} & \frac{11}{8} \\ \frac{1}{16} & \frac{7}{16} \end{bmatrix}$$

b IF $\mathbf{YA} + \mathbf{B} = \mathbf{C}$ then $\mathbf{YA} = \mathbf{C} - \mathbf{B}$

$$\therefore \mathbf{YA} = \begin{bmatrix} 3 & 4 \\ 2 & 6 \end{bmatrix} - \begin{bmatrix} 4 & -1 \\ 2 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} -1 & 5 \\ 0 & 4 \end{bmatrix}$$

From part **a**, $\mathbf{A}^{-1}=\begin{bmatrix}\frac{3}{8} & -\frac{1}{8}\\ -\frac{1}{16} & \frac{3}{16}\end{bmatrix}$

If $\mathbf{YA}=\begin{bmatrix}-1 & 5\\ 0 & 4\end{bmatrix}$, multiply both sides from the right by $\mathbf{A}^{-1}$.

$$\mathbf{YAA}^{-1}=\begin{bmatrix}-1 & 5\\ 0 & 4\end{bmatrix}\mathbf{A}^{-1}$$

$$\mathbf{YI}=\mathbf{Y}$$

$$=\begin{bmatrix}-1 & 5\\ 0 & 4\end{bmatrix}\begin{bmatrix}\frac{3}{8} & -\frac{1}{8}\\ -\frac{1}{16} & \frac{3}{16}\end{bmatrix}$$

$$=\begin{bmatrix}-1\times\frac{3}{8}+5\times-\frac{1}{16} & -1\times-\frac{1}{8}+5\times\frac{3}{16}\\ 0\times\frac{3}{8}+4\times-\frac{1}{16} & 0\times-\frac{1}{8}+4\times\frac{3}{16}\end{bmatrix}$$

$$\mathbf{X}=\begin{bmatrix}-\frac{11}{16} & \frac{17}{16}\\ -\frac{1}{4} & \frac{3}{4}\end{bmatrix}$$

7 $\mathbf{A}$ must be $\begin{bmatrix}a_{11} & 0\\ 0 & a_{22}\end{bmatrix}$.

$\det(\mathbf{A})=a_{11}\times a_{22}-0\times 0=a_{11}a_{22}$

$\det(\mathbf{A})\neq 0$ since $a_{11}\neq 0$ and $a_{22}\neq 0$ and

the product of two non-zero numbers cannot be zero.

$\therefore$ $\mathbf{A}$ is regular.

$$\mathbf{A}^{-1}=\frac{1}{a_{11}a_{22}}\begin{bmatrix}a_{22} & 0\\ 0 & a_{11}\end{bmatrix}$$

$$=\begin{bmatrix}\frac{1}{a_{11}} & 0\\ 0 & \frac{1}{a_{22}}\end{bmatrix}$$

8 If $\mathbf{A}$ is invertible, it will have an inverse, $\mathbf{A}^{-1}$. Multiply both sides of the equation $\mathbf{AB}=\mathbf{0}$ from the left by $\mathbf{A}^{-1}$.

$$\mathbf{A}^{-1}\mathbf{AB}=\mathbf{A}^{-1}0$$

$$\therefore \mathbf{IB}=0$$

$$\mathbf{B}=0$$

9 Let $\mathbf{A}$ be any matrix $\begin{bmatrix}a & b\\ c & d\end{bmatrix}$.

If the determinant is n, then the inverse of $\mathbf{A}$ is given by $\frac{1}{n}\begin{bmatrix}d & -b\\ -c & a\end{bmatrix}$.

$$\therefore \begin{bmatrix}a & b\\ c & d\end{bmatrix}=\frac{1}{n}\begin{bmatrix}d & -b\\ -c & a\end{bmatrix}$$

$$a=\frac{d}{n}\text{ and }d=\frac{a}{n}$$

Substituting for d, $a=\frac{a\div n}{n}=\frac{a}{n^2}$

This gives $n^2=1$, or $n=\pm1$.

If $n=1$, $a=d$ and $-b=b$, which gives $b=0$ and similarly $c=0$.

$\det(\mathbf{A})=ad=a^2=1$

This leads to two matrices, $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ and $\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$.

If $n = -1$, $a = -d$; there are no restrictions on b and c but the $\text{determinant} = ad - bc = -1$.

$\therefore a^2 + bc = 1 \quad (\text{since } a = -d)$

If $b = 0$, $a = \pm 1$, giving $\begin{bmatrix} \pm 1 & 0 \\ c & \mp 1 \end{bmatrix}$, which can be written $\begin{bmatrix} 1 & 0 \\ k & -1 \end{bmatrix}$ or $\begin{bmatrix} -1 & 0 \\ k & 1 \end{bmatrix}$.

If $b \neq 0,\ a^2 + bc = 1$ gives $c = \dfrac{1 - a^2}{b}$, giving $\begin{bmatrix} a & b \\ \dfrac{1 - a^2}{b} & -a \end{bmatrix}$, which includes the cases $\begin{bmatrix} 1 & k \\ 0 & -1 \end{bmatrix}$ and $\begin{bmatrix} -1 & k \\ 0 & 1 \end{bmatrix}$ when $a = \pm 1$.

10 $a = \pm\sqrt{2}$

11
$$\det(\mathbf{A}) = \frac{1}{n^2 + 2n} - \frac{1}{n^2 + 2n + 1}$$
$$= \frac{1}{n(n+1)^2(n+2)}$$

Therefore

$$\mathbf{A}^{-1} = n(n+1)^2(n+2)\begin{bmatrix} \frac{1}{n+2} & -\frac{1}{n} \\ -\frac{1}{n+1} & \frac{1}{n} \end{bmatrix}$$
$$= \begin{bmatrix} n(n+1)^2 & -(n+1)^2(n+2) \\ -n(n+1)(n+2) & (n+1)^2(n+2) \end{bmatrix}$$

All the entries are integers